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Equivariant splittings of supermanifolds

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Let G be a group acting on a supermanifold $(M, \mathcal{A})$. We ask whether $(M, \mathcal{A})$ is equivariantly isomorphic to the exterior sheaf of a vector bundle. When G is a Lie group and the underlying vector bundle of $\mathcal{A}$ is G-homogeneous, the question can be reduced to a finite-dimensional group-cohomology calculation. In particular, we obtain a vanishing theorem for the obstruction to an equivariant isomorphism, in case the isotropy subgroup is compact or semisimple.

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1. Let M be a C^∞ manifold and $\mathcal{E}\to M$ a locally free sheaf of C^∞_M-modules, i.e., the sheaf of sections of a smooth vector bundle. Then $\wedge\mathcal{E}\to M$ is a sheaf of supercommutative $\mathbb{R}$-algebras, in fact, a supermanifold. It is well known that every C^∞ supermanifold is of this form [B]. On the other hand, let $\mathrm{Aut}(\wedge\mathcal{E})$ be the group of automorphisms of $\wedge\mathcal{E}$. By definition, an element of $\mathrm{Aut}(\wedge\mathcal{E})$ is a diffeomorphism $M\overset{\phi_{\mathrm{red}}}{\approx}M$ together with an isomorphism $\phi^{-1}_{\mathrm{red}}(\wedge\mathcal{E})\overset{\phi^*}{\to}\wedge\mathcal{E}$ of sheaves, respecting the $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra structure. One similarly defines $\mathrm{Aut}(\mathcal{E})$ to be the group whose elements are pairs $M\overset{\phi_{\mathrm{red}}}{\approx}M$ together with an isomorphism $\phi^{-1}_{\mathrm{red}}(\mathcal{E})\overset{\phi^*}{\approx}\mathcal{E}$ of sheaves, respecting the C^∞_M-module structure. There is an obvious injection $\mathrm{Aut}(\mathcal{E})\overset{\iota}{\rightarrowtail}\mathrm{Aut}(\wedge\mathcal{E})$ associating to any $\phi\in\mathrm{Aut}(\mathcal{E})$ the unique element of $\mathrm{Aut}(\wedge\mathcal{E})$ which extends ϕ *and preserves the $\mathbb{Z}$-grading of* $\wedge\mathcal{E}$. In fact the image of ι consists precisely of those automorphisms which preserve the $\mathbb{Z}$-grading. There are additional automorphisms of $\wedge\mathcal{E}$ obtained as follows. Let $\mathrm{Der}(\wedge\mathcal{E})$ denote the Lie superalgebra of derivations of $\wedge\mathcal{E}$. By definition, an element of $\mathrm{Der}(\wedge\mathcal{E})$ is a function associating to each open set $U\subset M$ a superderivation of $\wedge\mathcal{E}(U)$. Then let $\mathrm{Der}^+(\wedge\mathcal{E})$ denote the Lie subalgebra of $\mathrm{Der}(\wedge\mathcal{E})$ consisting of those

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derivations which increase degree and preserve the $\mathbb{Z}_2$-grading. For any $Y \in \mathrm{Der}^+(\wedge\mathscr{E})$, e^Y is well defined as a polynomial in Y, so we obtain a subgroup $N(\wedge\mathscr{E}) = \exp(\mathrm{Der}^+(\wedge\mathscr{E})) \subset \mathrm{Aut}(\wedge\mathscr{E})$. The image of ι intersects $N(\wedge\mathscr{E})$ trivially. In fact we have

Theorem 1 (cf. ref. [R]). *There is an exact sequence*

$$N(\wedge\mathscr{E}) \rightarrowtail \mathrm{Aut}(\wedge\mathscr{E}) \twoheadrightarrow \mathrm{Aut}(\mathscr{E}), \qquad \phi \rightsquigarrow \tilde{\phi},$$

defined by the formulas $\tilde{\phi}_{\mathrm{red}} = \phi_{\mathrm{red}}$, $\tilde{\phi}^*(\alpha) \equiv \phi^*(\alpha) \bmod \mathscr{N}^2$. *Since ι splits this sequence, we have*

$$\mathrm{Aut}(\wedge\mathscr{E}) = \mathrm{Aut}(\mathscr{E}) \ltimes N(\wedge\mathscr{E}).$$

Proof. The map $\phi \rightsquigarrow \tilde{\phi}$ is clearly surjective, provided it is well defined. For this it suffices to show that for all $\phi \in \mathrm{Aut}(\wedge\mathscr{E})$ and all $f \in C^\infty_M = \wedge^0\mathscr{E}$, $(\phi^*(f))_0 = f \circ \phi_{\mathrm{red}}$, where $(\phi^*(f))_0$ is the degree-0 component of $\phi^*(f)$. But for all $p \in M$, the map $C^\infty_M|_{\phi(p)} \to \mathbb{R}$, $f \rightsquigarrow (\phi^*(f))_0(p)$, defines an $\mathbb{R}$-linear homomorphism, and there is only one such, namely $f \rightsquigarrow f(\phi(p))$. Now suppose $\phi \in \mathrm{Aut}(\wedge\mathscr{E})$ and $\tilde{\phi}$ is trivial. We must show that $\phi \in N(\wedge\mathscr{E})$. Since $1-\phi$ induces the identity on $\wedge\mathscr{E}/\mathscr{N}^2$, and since ϕ is an automorphism, it follows that $1-\phi$ is degree increasing. Then we may define $Y = \log(\phi) = -\sum(1-\phi)^n/n$, the sum being finite. Then $Y \in \mathrm{Der}^+(\wedge\mathscr{E})$ and $\phi = e^Y$. □

Now let G be a group acting on $\wedge\mathscr{E}$. I.e., consider a homomorphism $G \xrightarrow{T} \mathrm{Aut}(\wedge\mathscr{E})$. We get a homomorphism $G \xrightarrow{\tilde{T}} \mathrm{Aut}(\mathscr{E})$, $\tilde{T}(g) = \widetilde{T(g)}$. On the other hand, the inclusion $\mathrm{Aut}(\mathscr{E}) \rightarrowtail \mathrm{Aut}(\wedge\mathscr{E})$ allows us to regard $\tilde{T}$ as another action of G on $\wedge\mathscr{E}$ (which we continue to call $\tilde{T}$). The *equivariant* splitting problem is to determine whether T and $\tilde{T}$ are equivalent, in the sense that they are intertwined by conjugation by some element of $\mathrm{Aut}(\wedge\mathscr{E})$. The source of this terminology is the following. The short exact sequence

$$0 \to \mathscr{N}^2 \to \wedge\mathscr{E} \to \mathscr{E} \oplus C^\infty_M \to 0 \qquad (*)$$

is $\mathrm{Aut}(\wedge\mathscr{E})$-equivariant, and therefore G-equivariant. Then T and $\tilde{T}$ are equivalent if and only if $(*)$ splits G-equivariantly as a sequence of sheaves of $\mathbb{Z}_2$-graded algebras. Note also that $(*)$ splits equivariantly if and only if the $\mathbb{Z}$-grading on $\wedge\mathscr{E}$ can be redefined in such a way that it is preserved by G. Indeed, $\tilde{T}$ certainly preserves the standard $\mathbb{Z}$-grading, which gives one direction of this assertion. Conversely, if $\oplus_{i=0}^{rk\mathscr{E}}\mathscr{A}_i$ is a $\mathbb{Z}$-grading preserved by $T(G)$, then $(*)$ gives $\mathscr{A}_0 \oplus \mathscr{A}_1 \approx C^\infty_M \oplus \mathscr{E}$. The inverse of this isomorphism extends to an automorphism $\phi \in \mathrm{Aut}(\wedge\mathscr{E})$. Then ϕ intertwines T and $\tilde{T}$. Note that ϕ necessarily belongs to $N(\wedge\mathscr{E})$. Another way of thinking about the equivariant splitting problem is to imagine the supermanifold $(M, \mathscr{A})$ given without an explicit isomorphism with

$\wedge \mathscr{E}$, but with a group action. Then we are asking whether $\mathscr{A}$ and $\wedge \mathscr{E}$ are G-equivariantly isomorphic.

We will solve the equivariant splitting problem in the case that G is a Lie group, $M = G/H$ for some closed subgroup H, and $\mathscr{E}$ is the sheaf of sections $G \times_H V$ for some finite-dimensional H-module V. By solving we mean that the problem will be reduced to a calculation in the first cohomology group of H with coefficients in an appropriate *finite-dimensional* H-module W. The reduction is accomplished by means of Shapiro's lemma [BW], which asserts the equality of the H-cohomology groups of W and the G-cohomology groups of $\mathrm{Ind}_H^G(W)$.

2. We begin by translating the question of equivariant splitting into a cohomology problem. Let $r = \mathrm{rank}(\mathscr{E})$. For each $g \in G$, we may use theorem 1 to define $Y(g) \in \mathrm{Der}^+(\wedge \mathscr{E})$ by $T(g) = \mathrm{e}^{Y(g)} \tilde{T}(g)$. For any integer $j \leq r$, let $\mathrm{Der}^j(\wedge \mathscr{E})$ denote the jth graded piece of $\mathrm{Der}(\wedge \mathscr{E})$, and let $Y_j(g)$ denote the piece of $Y(g)$ belonging to $\mathrm{Der}^j(\wedge \mathscr{E})$. Let us say $Y(g)$ vanishes to order j if $Y_k(g) = 0$ for all $k \leq j$. If we conjugate T by some element $\mathrm{e}^Z \in N(\wedge \mathscr{E})$, it may be possible to increase the order of Y. We define the order of T to be the supremum of integers $j \leq r$ such that there exists $Z \in \mathrm{Der}^+(\wedge \mathscr{E})$ such that for all $g \in G$, $\mathrm{e}^Z T(g) \mathrm{e}^{-Z} = \mathrm{e}^{Y'(g)} \tilde{T}(g)$, where $Y'(g)$ vanishes to order j. Clearly, T and $\tilde{T}$ are equivalent if and only if T has order r. Suppose then that T has order j, and assume, after conjugating T appropriately, that for all $g \in G$, Y vanishes to order j. Then set $Y(g) = Y_{j+1}(g) + \text{higher order terms}$.

Theorem 2. *The function $g \rightsquigarrow Y_{j+1}(g)$ is a cocycle with respect to the adjoint action of $\tilde{T}(G)$ on $\mathrm{Der}^{j+1}(\wedge \mathscr{E})$, i.e.,*

$$Y_{j+1}(gh) = Y_{j+1}(g) + \tilde{T}(g) Y_{j+1}(h) \tilde{T}(g^{-1}) .$$

Its cohomology class is an invariant of T, and is non-zero if and only if T and $\tilde{T}$ are equivalent.

Proof. For all $g, h \in G$,

$$(1 + Y_{j+1}(g)) \tilde{T}(g) (1 + Y_{j+1}(h)) \tilde{T}(h) \equiv (1 + Y_{j+1}(gh)) \tilde{T}(gh)$$

modulo terms of order $j+2$. That is,

$$Y_{j+1}(g) \tilde{T}(gh) + \tilde{T}(g) Y_{j+1}(h) \tilde{T}(h) = Y_{j+1}(gh) \tilde{T}(gh) ,$$

or equivalently,

$$Y_{j+1}(gh) = Y_{j+1}(g) + \tilde{T}(g) Y_{j+1}(h) \tilde{T}(g^{-1}) .$$

This proves the first assertion. Now suppose we conjugate T by an element of $\phi \in N(\wedge \mathscr{E})$ in such a way that for all $g \in G$, the order of $Y(g)$ does not decrease. We may write ϕ in the form $\mathrm{e}^{Z_r} \cdots \mathrm{e}^{Z_2}$, $\mathrm{e}^{Z_k} \in \mathrm{Der}^k(\wedge \mathscr{E})$. It follows by induction on

k that Z_k commutes with $\tilde{T}(g)$ for $k \leqslant j$, and therefore $Y_{j+1}(g)$ changes by

$$Y_{j+1}(g) \rightsquigarrow Y_{j+1}(g) + Z_{j+1} - \tilde{T}(g) Z_{j+1} \tilde{T}(g^{-1}) ,$$

i.e., by a coboundary. This proves the second assertion. The third assertion follows from the very definition of Y_{j+1}. □

3. Specialize now to the case described in section 1: G is a real analytic group, H is a closed subgroup, $M = G/H$, $H \to^{\rho} \mathrm{GL}(V)$ is a finite-dimensional representation of H, and $\mathscr{E}$ is the sheaf of C^∞ sections of the homogeneous vector bundle $G \times_\rho V$. Let $\mathfrak{h}$ denote the Lie algebra of H, and $\dot{\rho}: \mathfrak{h} \to \mathrm{End}(V)$ the derivative of ρ. Then we have $\mathrm{graph}(\dot{\rho}) \subset \mathrm{End}\ V \oplus \mathfrak{h}$. Each element of End V extends to derivation of $\wedge V$, so $\mathrm{graph}(\dot{\rho})$ may be regarded as an H-submodule of $\mathrm{Der}(\wedge V) \oplus \mathfrak{g}$, with H acting on $\mathfrak{g}$ by Ad. Note that $\mathrm{Der}(\wedge V) = \wedge V \otimes V^*$. Let $\langle \mathrm{graph}(\dot{\rho}) \rangle$ denote the $\wedge V$-submodule of $\wedge V \otimes (V^* \oplus \mathfrak{g})$ generated by $\mathrm{graph}(\dot{\rho})$. Set $W = (\wedge V \otimes (V^* \oplus \mathfrak{g})) / \langle \mathrm{graph}(\dot{\rho}) \rangle$.

Proposition 3. $\mathrm{Der}(\wedge \mathscr{E}) = \mathrm{Ind}_H^G(W)$, *i.e. the space of sections of* $G \times_H W$.

Proof. Clearly, sections of $G \times_H (\wedge V \otimes V^*)$ determine derivations of $\wedge \mathscr{E}$. So do sections of $G \times_H \mathfrak{g}$, for, if $\varphi: G \to \mathfrak{g}$ transforms under H by Ad and $f: G \to \wedge V$ transforms under H by ρ, then for all $a \in G$, define

$$\varphi f(a) = (\mathrm{d}/\mathrm{d}t)|_0 f(a \mathrm{e}^{t\varphi(a)}) .$$

Then for all $b \in H$,

$$\begin{aligned}\varphi f(ab) &= (\mathrm{d}/\mathrm{d}t)|_0 f(ab\mathrm{e}^{t\varphi(ab)}) \\ &= (\mathrm{d}/\mathrm{d}t)|_0 f(a\mathrm{e}^{t\varphi(a)} b) \\ &= \rho(b^{-1}) \varphi f(a) .\end{aligned}$$

It is easy to see then that every derivation comes from a section of $G \times_H (\wedge V \otimes V^* \oplus \wedge V \otimes \mathfrak{g})$, and that the kernel is the space of sections of $G \times_H \langle \mathrm{graph}(\dot{\rho}) \rangle$. □

Recall from ref. [HM] the notion of a differentiable G-module. If A is a real topological vector space, and also a G-module, then it is a *continuous* G-module if the map $G \times A \to A$, $(g, a) \rightsquigarrow ga$ is continuous. If, furthermore, for all $a \in A$, the map $G \to A$, $g \rightsquigarrow g \cdot a$ is an infinitely differentiable map, then A is called a *differentiable* G-module. If B is any real topological vector space whose points are separated by continuous linear functionals, then we get a differentiable G-module by putting a suitable topology on the space of all differentiable maps from G to B.

In particular, if U is any finite-dimensional H-module, and $\mathscr{F}$ is the sheaf of C^∞ sections on $G \times_H U$, then $\mathrm{C}^\infty(G) \otimes U$ is a differentiable G-module, and therefore so is the sub-module $\Gamma(M, \mathscr{F})$. Thus it makes sense to consider the differentiable cohomology groups, $H^*_{\rm d}(G, \Gamma(M, \mathscr{F}))$. We can of course also consider the differentiable cohomology groups $H^*_{\rm d}(H, U)$. We then have

Shapiro's lemma. *$H^*_{\rm d}(G, \Gamma(M, \mathscr{F}))$ and $H^*_{\rm d}(H, U)$ are canonically isomorphic.*

Proof. For the details of the proof, see ref. [BW]. Since we will be using the isomorphism in degree 1 explicitly in the example below, we give a sketch of the construction. Make $C^\infty(G)\otimes U$ a $G\times H$-module, letting G act by $l_g(f\otimes u)(g')=f(g^{-1}g')\otimes u$, and letting H act by ${}^h(f\otimes u)(g)=f(gh)\otimes h\cdot u$. Then U consists precisely of the G-invariants, and $\Gamma(M, \mathscr{F})$ consists precisely of the H-invariants. Moreover, $C^\infty(G)\otimes U$ turns out to be acyclic for both G and H. Then Shapiro's lemma follows from a standard spectral sequence argument. In degree 1, the isomorphism works as follows. Let $\omega\in H^1_{\rm d}(H, U)$ be represented by a cocycle $c:H\to U$. Regarding c as a $C^\infty(G)\otimes U$-valued cocycle, we may choose some $f\in C^\infty(G)\otimes U$ such that for all $h\in H$, $c(h)={}^hf-f$. Then regarding f as a zero-chain with respect to G, we take its coboundary, $\delta f(g)(g')=f(g^{-1}g')-f(g')$. Then the isomorphism $H^1_{\rm d}(H, U)\approx H^1_{\rm d}(G, \Gamma(M, \mathscr{F}))$ takes ω to the class of δf. □

In combination with Shapiro's lemma, theorem 2 implies

Theorem 4. *With the notation as above, assume the function $g\rightsquigarrow Y(g)$ is differentiable. Then we may canonically associate to T an integer $j\leqslant \mathrm{Dim}(V)$ and a cohomology class $\omega\in H^1_{\rm d}(H, W_{j+1})$, where W_{j+1} is the $(j+1)$st component of W, such that ω vanishes if and only if $j=\mathrm{Dim}(V)$ if and only if T and $\tilde{T}$ are equivalent.* □

By theorem 4, we can deduce the equivalence of T and $\tilde{T}$ in many cases without any computation. For example,

Corollary 5. *With the same assumptions as in theorem 4, suppose in addition that H is compact or semisimple. Then T and $\tilde{T}$ are equivalent.*

Proof. In this case, $H^1_{\rm d}(H, U)$ vanishes for any finite-dimensional H-module U. For the convenience of the reader, we recall the proof of this well-known fact. If $c:H\to U$ is any cocycle, then let $M=U^*\oplus\mathbb{C}$ with the H-module structure

$$h\cdot(\alpha, z)=((h^{-1})^*\alpha, \langle c(h), \alpha\rangle+z) ,$$

where $\langle\ ,\ \rangle$ denotes the pairing between U and U^*. The short exact sequence

$$0\to\mathbb{C}\to M\to U^*\to 0$$

must split as a sequence of H-modules. This gives a linear map from U^* to $\mathbb{C}$, i.e. an element $u \in U$. Then c is the coboundary of u. □

To produce a nonsplittable action of G, we must choose $H^1(H, W) \neq 0$. The easiest way to get an example is to find a class $\omega \in H^1(G, \mathrm{Der}^j(\wedge \mathscr{E}))$ where $j > rk\mathscr{E}/2$. Then if we represent ω by a cocycle $G \xrightarrow{\zeta} \mathrm{Der}^j(\wedge \mathscr{E})$, we get a family of deformed actions of G:

$$g \rightsquigarrow \exp(t\zeta(g))l_g = (1 + t\zeta(g))l_g ,$$

where we have written l_g to emphasize that initially g acts by the left regular representation on $\Gamma(M, \wedge \mathscr{E})$.

Example. Let

$$G = \mathrm{SL}(2, \mathbb{R}), \qquad N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\} .$$

Then $G/N = \mathbb{R}^2 - \{0\}$. Let E be the trivial bundle of rank 2 on G/N, with basis e_1, e_2. Then elements of $\mathrm{Der}^2(\wedge \mathscr{E})$ are of the form $e_1 \wedge e_2 Z$, where Z is simply a vector field on $\mathbb{R}^2 - \{0\}$. Let L denote the space of smooth vector fields on $\mathbb{R}^2 - \{0\}$. By Shapiro's lemma,

$$H^1_{\mathrm{d}}(SL(2, \mathbb{R}), L) \approx H^1_{\mathrm{d}}(N, \mathfrak{g}/\mathfrak{n}) ,$$

where $\mathfrak{g}$ and $\mathfrak{n}$ are the Lie algebras of G and N, respectively. Let us use this isomorphism to produce a nontrivial cocycle $\mathrm{SL}(2, \mathbb{R}) \xrightarrow{Z} L$. Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

If we take H, Y to represent a basis for $\mathfrak{g}/\mathfrak{n}$, we find that the action of N on $\mathfrak{g}/\mathfrak{n}$ is equivalent to its defining action on $\mathbb{R}^2$. Identify N with $\mathbb{R}$. Thus a $\mathfrak{g}/\mathfrak{n}$-valued cocycle becomes a smooth map

$$\mathbb{R} \xrightarrow{c} \mathbb{R}^2$$

such that for all s, $t \in \mathbb{R}$,

$$c(s+t) = c(s) + \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} c(t) .$$

Writing

$$c(s) = \begin{pmatrix} a(s) \\ b(s) \end{pmatrix} ,$$

we find that the general smooth solution is $b(s) = b_0 s$, $a(s) = a_0 s + \frac{1}{2} b_0 s^2$. Coboundaries are functions of the form

$$c(s)=v-\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}v, \quad \text{for some } v\in\mathbb{R}^2 ,$$

i.e.,

$$c(s)=\begin{pmatrix}\lambda s\\ 0\end{pmatrix}, \quad \lambda\in\mathbb{R} .$$

Therefore we may assume $a_0=0$, and thus $H^1_{\mathrm{d}}(N, \mathfrak{g}/\mathfrak{n})$ is spanned by the cocycle $c(s)\equiv s^2H+2sY \pmod X$. To trace through Shapiro's lemma, we must regard c as a cocycle with coefficients in $C^\infty(G)\otimes\mathfrak{g}/\mathfrak{n}$ and then find a function $W: G\to \mathfrak{g}/\mathfrak{n}$ such that c is the coboundary of W. The general element of G may be written

$$h=\begin{pmatrix}\lambda\cos\theta & s\lambda\cos\theta-\lambda^{-1}\sin\theta\\ \lambda\sin\theta & s\lambda\sin\theta+\lambda^{-1}\cos\theta\end{pmatrix}=h(\theta,\lambda,s) .$$

Thus we may seek a W which depends only on s, and we readily find the solution $W(s)=s^2H-2sY$. Indeed, for all s, t,

$$\begin{pmatrix}t^2\\ -2t\end{pmatrix}-\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\begin{pmatrix}(t+s)^2\\ -2(t+s)\end{pmatrix}=\begin{pmatrix}t^2-s^2-2st-t^2+2s^2+2st\\ -2t+2s+2t\end{pmatrix}=\begin{pmatrix}s^2\\ 2s\end{pmatrix}.$$

Now let δW be the coboundary of W with respect to G. Thus, for all $g\in G$, $\delta W(g)$ is the function $\delta W(g): G\to\mathfrak{g}/\mathfrak{n}$ defined by

$$\delta W(g)(h)=W(s(h))-W(s(g^{-1}h)) .$$

We are to think of $\delta W(g)$ as a vector field on $\mathbb{R}^2-\{0\}$. To write this out explicitly, first write the left invariant vector fields H_{L} and Y_{L} in terms of $\partial/\partial\theta$, $\partial/\partial\lambda$ and $\partial/\partial s$, and mod out the $\partial/\partial s$ term. The result is

$$H_{\mathrm{L}}\equiv\lambda\frac{\partial}{\partial\lambda}\bmod\frac{\partial}{\partial s},$$

$$Y_{\mathrm{L}}\equiv\frac{1}{\lambda^2}\frac{\partial}{\partial\theta}+s\lambda\frac{\partial}{\partial\lambda}\bmod\frac{\partial}{\partial s}.$$

Now define $\varphi_g(h)=s(h)-s(g^{-1}h)$. We now have

$$\delta W(g)(h)\equiv[s(h)^2-s(g^{-1}h)^2]H_{\mathrm{L}} \\ -2[s(h)-s(g^{-1}h)]Y_{\mathrm{L}}\bmod\partial/\partial s ,$$

which yields

$$\delta W(g)(h) = [s(h)^2 - s(g^{-1}h)^2]\lambda \frac{\partial}{\partial\lambda} - 2[s(h) - s(g^{-1}h)]\left(s\lambda\frac{\partial}{\partial\lambda} + \frac{1}{\lambda^2}\frac{\partial}{\partial\theta}\right)$$

$$= \varphi_g(\theta, \lambda)[s(h) + s(g^{-1}h)]\lambda\frac{\partial}{\partial\lambda} - 2\varphi_g(\theta,\lambda)\left(s(h)\lambda\frac{\partial}{\partial\lambda} + \frac{1}{\lambda^2}\frac{\partial}{\partial\theta}\right)$$

$$= -\varphi_g(h)^2\lambda\frac{\partial}{\partial\lambda} - 2\varphi_g(h)\frac{1}{\lambda^2}\frac{\partial}{\partial\theta}.$$

To be completely explicit, let us write out φ_g as a function of θ and λ. If $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$s(h) = \frac{b}{a} + \frac{a}{c(a^2+c^2)}.$$

Set

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

If we assume, as we may, that $s(h) = 0$, then

$$\varphi_g(\theta, \lambda) = -s(g^{-1}h)$$

$$= -s\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \lambda\cos\theta & -\lambda^{-1}\sin\theta \\ \lambda\sin\theta & \lambda^{-1}\cos\theta \end{pmatrix}\right)$$

$$= \frac{\alpha\lambda^{-1}\sin\theta - \beta\lambda^{-1}\cos\theta}{\alpha\lambda\cos\theta + \beta\lambda\sin\theta} - \frac{\gamma\lambda\cos\theta + \delta\lambda\sin\theta}{\alpha\lambda\cos\theta + \beta\lambda\sin\theta}$$

$$\times \frac{1}{(\alpha\lambda\cos\theta + \beta\lambda\sin\theta)^2 + (\gamma\lambda\cos\theta + \delta\lambda\sin\theta)^2}.$$

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